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Technical Notes

Improvements to Obtain a Unique Solution in System Identification

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Nomenclature

$\boldsymbol{A}$	=	diagonal matrix with arbitrary constant as its elements
$\boldsymbol{E}$	=	unit matrix
$\boldsymbol{K}$	=	identified stiffness matrix
$\boldsymbol{K}_A$	=	stiffness matrix of finite element model
$\boldsymbol{M}$	=	identified mass matrix
$\boldsymbol{M}_A$	=	mass matrix of finite element model
n_A	=	total number of analytical modes
n_T	=	total number of measured modes
$\boldsymbol{P}$	=	matrix defined by Eq. (3)
α, β	=	Lagrange multipliers
$\Delta\boldsymbol{K}$	=	difference between $\boldsymbol{K}$ and $\boldsymbol{K}_A$
$\Delta\boldsymbol{M}$	=	difference between $\boldsymbol{M}$ and $\boldsymbol{M}_A$
$\boldsymbol{\phi}$	=	modal matrix
$\boldsymbol{\Omega}^2$	=	measured frequency matrix

I. Introduction

FREQUENCY is one of the important design requirements for space structures. Finite element analysis is a powerful tool to find dynamic characteristics for space structures. Finite element models of those structures will need a greater degree of freedom as they become larger and more complex. In such cases, it is very difficult to obtain accurate solutions due to modeling errors on initial finite element models. System identification is a key method in predicting the accurate dynamic behavior of structures. Several types of methods [1–6] have been proposed to establish the analytical models that correspond with modal test results. One of these methods assumes an initial, undamped, finite element model which is characterized by its mass and stiffness matrices [3–6]. This method corrects the mass and stiffness matrices by minimizing the Euclidean norm subject to constraints. These constraints include modal mass, modal stiffness, mode orthogonality, the symmetry of mass and stiffness matrices, the dynamic equation, and the connectivity of finite elements [7]. The mass matrix is not uniquely identified in these methods because modal masses are considered in the constraints. Since the mode shapes are not uniquely defined, any mode shape can be multiplied by a nonzero constant without changing its physical meaning. Therefore, modal mass constraints are used to normalize mode shapes. However, the identified mass matrix depends on nonzero constants when using modal mass constraints in mass matrix identification. As a result, there exist an infinite number of identified mass matrices that satisfy the constraints.

The purpose of this Note is to show improvements upon existing identification methods using constrained minimization theory for identifying a unique mass matrix.

II. Cause of Nonunique Solutions in Mass Matrix Identification

In this section, the cause that the identified mass matrix is not unique is clarified to identify the unique mass matrix.

We deal with system identification methods using the constrained minimization theory. These methods have the potential to update finite element models with greater degrees of freedom. All the methods identify mass and stiffness matrices by minimizing an objective function under several constraints. A typical method [4] using the constrained minimization theory is described below. In this method, the modal matrix satisfies the modal mass and mode orthogonal conditions:

$$\boldsymbol{\phi}^T(\boldsymbol{M}_A + \Delta\boldsymbol{M})\boldsymbol{\phi} = \boldsymbol{E} \tag{1}$$

Equation (1) can be rewritten as

$$\boldsymbol{\phi}^T\Delta\boldsymbol{M}\boldsymbol{\phi} = \boldsymbol{E} - \boldsymbol{P} \tag{2}$$

where

$$\boldsymbol{P} = \boldsymbol{\phi}^T\boldsymbol{M}_A\boldsymbol{\phi} \tag{3}$$

The mass matrix is identified by minimizing the Euclidean norm $\|\boldsymbol{M} - \boldsymbol{M}_A\|$ subject to the modal mass, mode orthogonal, and symmetry constraints. Using the constrained minimization theory, the Lagrange function is expressed by

$$\Psi = \|\Delta\boldsymbol{M}\| + \sum_{i=1}^{n_T}\sum_{j=1}^{n_T}\alpha_{ij}(\boldsymbol{\phi}^T\Delta\boldsymbol{M}\boldsymbol{\phi} - \boldsymbol{E} + \boldsymbol{P})_{ij} + \sum_{i=1}^{n_A}\sum_{j=1}^{n_A}\beta_{ij}(\boldsymbol{M} - \boldsymbol{M}^T)_{ij} \tag{4}$$

where

$$\Delta\boldsymbol{M} = \boldsymbol{M} - \boldsymbol{M}_A \tag{5}$$

When Ψ represents the minimum, the optimum solution $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \boldsymbol{M}_A + \boldsymbol{M}_A\boldsymbol{\phi}\boldsymbol{P}^{-1}(\boldsymbol{E} - \boldsymbol{P})\boldsymbol{P}^{-1}\boldsymbol{\phi}^T\boldsymbol{M}_A \tag{6}$$

The problem is that the identified mass matrix is not unique. Since the mode shape is not uniquely defined, any mode shape can be multiplied by a nonzero constant without changing its physical meaning:

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$$\bar{\boldsymbol{\phi}} = \boldsymbol{\phi} \boldsymbol{A} \tag{7}$$

If we assume that another solution $\Delta\bar{\boldsymbol{M}}$ satisfying Eq. (2) exists, the following equation is obtained:

$$\bar{\boldsymbol{\phi}}^T \Delta\bar{\boldsymbol{M}} \bar{\boldsymbol{\phi}} = \boldsymbol{E} - \bar{\boldsymbol{P}} \tag{8}$$

where

$$\bar{\boldsymbol{P}} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A} \tag{9}$$

Using Eq. (6), we obtain the optimum solution $\bar{\boldsymbol{M}}$ as

$$\bar{\boldsymbol{M}} = \boldsymbol{M}_A + \boldsymbol{M}_A \bar{\boldsymbol{\phi}} \bar{\boldsymbol{P}}^{-1} (\boldsymbol{E} - \bar{\boldsymbol{P}}) \bar{\boldsymbol{P}}^{-1} \bar{\boldsymbol{\phi}}^T \boldsymbol{M}_A \tag{10}$$

Substituting Eqs. (7) and (9) into Eq. (10), we obtain

$$\bar{\boldsymbol{M}} = \boldsymbol{M}_A + \boldsymbol{M}_A \boldsymbol{\phi} \boldsymbol{P}^{-1} (\boldsymbol{A}^{-2} - \boldsymbol{P}) \boldsymbol{P}^{-1} \boldsymbol{\phi}^T \boldsymbol{M}_A \tag{11}$$

$$\left.\begin{array}{r} \sum_{i=1}^{n_A} \sum_{j=1}^{n_T} \tau_{ij} \boldsymbol{\phi}^T (\boldsymbol{D}_{ij} + \boldsymbol{D}_{ij}^T) \boldsymbol{\phi} - \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \nu_{ij} \boldsymbol{\phi}^T (\boldsymbol{H}_{ij} + \boldsymbol{H}_{ij}^T) \boldsymbol{\phi} = \boldsymbol{\phi}^T \boldsymbol{K}_A \boldsymbol{\phi} - \boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \\ \sum_{i=1}^{n_A} \sum_{j=1}^{n_T} \tau_{ij} (\boldsymbol{D}_{ij} + \boldsymbol{D}_{ij}^T) \boldsymbol{\phi} - \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \nu_{ij} (\boldsymbol{H}_{ij} + \boldsymbol{H}_{ij}^T) \boldsymbol{\phi} = \boldsymbol{K}_A \boldsymbol{\phi} - \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \end{array}\right\} \tag{16}$$

If another solution $\bar{\boldsymbol{K}}$ exists, the identified stiffness matrix is obtained as

$$\begin{array}{c} \bar{\boldsymbol{K}} = \boldsymbol{K}_A - \sum_{i=1}^{n_A} \sum_{j=1}^{n_T} \bar{\tau}_{ij} (\bar{\boldsymbol{D}}_{ij} + \bar{\boldsymbol{D}}_{ij}^T) - \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \bar{\nu}_{ij} (\bar{\boldsymbol{H}}_{ij} + \bar{\boldsymbol{H}}_{ij}^T) \\ \bar{\boldsymbol{K}} = \boldsymbol{K}_A - \sum_{i=1}^{n_A} \sum_{j=1}^{n_T} \bar{\tau}_{ij} a_j (\boldsymbol{D}_{ij} + \boldsymbol{D}_{ij}^T) - \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \bar{\nu}_{ij} a_i a_j (\boldsymbol{H}_{ij} + \boldsymbol{H}_{ij}^T) \end{array} \tag{17}$$

A simultaneous equation to find Lagrange multipliers $\bar{\tau}_{ij}$ and $\bar{\nu}_{ij}$ is expressed as

$$\left.\begin{array}{r} \sum_{i=1}^{n_A} \sum_{j=1}^{n_T} \bar{\tau}_{ij} a_j \boldsymbol{\phi}^T (\boldsymbol{D}_{ij} + \boldsymbol{D}_{ij}^T) \boldsymbol{\phi} - \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \bar{\nu}_{ij} a_i a_j \boldsymbol{\phi}^T (\boldsymbol{H}_{ij} + \boldsymbol{H}_{ij}^T) \boldsymbol{\phi} = \boldsymbol{\phi}^T \boldsymbol{K}_A \boldsymbol{\phi} - \boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \\ \sum_{i=1}^{n_A} \sum_{j=1}^{n_T} \bar{\tau}_{ij} a_j (\boldsymbol{D}_{ij} + \boldsymbol{D}_{ij}^T) \boldsymbol{\phi} - \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \bar{\nu}_{ij} a_i a_j (\boldsymbol{H}_{ij} + \boldsymbol{H}_{ij}^T) \boldsymbol{\phi} = \boldsymbol{K}_A \boldsymbol{\phi} - \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \end{array}\right\} \tag{18}$$

This equation shows that there exist an infinite number of identified mass matrices because all the elements of the diagonal matrix $\boldsymbol{A}$ are arbitrary. Baruch [8] also points out the same problem when all modes are considered (the modal matrix is square). This results in the use of modal masses as constraints. It should be noted that the mode shape is the ratio of amplitude at each point on a structure. Therefore, we can normalize the modes in several different expressions according to our purposes. Equation (1) is just one of these expressions. The modal mass constraints are not the constraint imposed on the mass matrix but on mode shapes. This implies that it is not reasonable to use the modal mass constraints for mass matrix identification. Therefore, we must exclude the modal mass constraints for identifying unique mass matrix.

III. Unique Solution in Stiffness Matrix Identification

The identified mass matrix can be obtained by excluding the modal mass constraints in mass matrix identification. In this section, it is shown that the identified stiffness matrix is unique under the preexisting condition that the identified mass matrix is unique.

The Lagrange function is given by the following equation for stiffness matrix identification:

$$\begin{aligned} \Psi = \|\Delta \boldsymbol{K}\| &+ \sum_{i=1}^{n_A} \sum_{j=1}^{n_A} \gamma_{ij} (\boldsymbol{K} - \boldsymbol{K}^T)_{ij} \\ &+ \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \tau_{ij} (\boldsymbol{\phi}^T \Delta \boldsymbol{K} \boldsymbol{\phi} - \boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 + \boldsymbol{\phi}^T \boldsymbol{K}_A \boldsymbol{\phi})_{ij} \\ &+ \sum_{i=1}^{n_A} \sum_{j=1}^{n_T} \nu_{ij} (\Delta \boldsymbol{K} \boldsymbol{\phi} - \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 + \boldsymbol{K}_A \boldsymbol{\phi})_{ij} \end{aligned} \tag{12}$$

The second term of the right-hand side of Eq. (12) is the symmetry requirements. The third term is the modal stiffness and mode orthogonal constraints, and the last term is the dynamic equation constraints. Differentiating Eq. (12) with respect to each element of $\boldsymbol{K}$ and equating the results to zero gives the optimal solution $\boldsymbol{K}$ as

$$\boldsymbol{K} = \boldsymbol{K}_A - \sum_{i=1}^{n_A} \sum_{j=1}^{n_T} \tau_{ij} (\boldsymbol{D}_{ij} + \boldsymbol{D}_{ij}^T) - \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \nu_{ij} (\boldsymbol{H}_{ij} + \boldsymbol{H}_{ij}^T) \tag{13}$$

where

$$\boldsymbol{D}_{ij} = \boldsymbol{C}_i \boldsymbol{\phi}_j^T, \qquad \boldsymbol{H}_{ij} = \boldsymbol{\phi}_i \boldsymbol{\phi}_j^T \tag{14}$$

Vector $\boldsymbol{C}_i$ is 1 at the ith row and 0 for the rest of the rows as

$$\boldsymbol{C}_i^T = \begin{array}{c} \phantom{\{0 \; 0 \; \cdots \; 0 \;} i } \\ \{0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad 0 \quad \cdots \quad 0\} \end{array} \tag{15}$$

Lagrange multipliers τ_{ij} and ν_{ij} are found by solving the following simultaneous equations.

Setting variables in Eq. (18) as $\bar{\tau}_{ij} a_j$ and $\bar{\nu}_{ij} a_i a_j$, we obtain the following relations from Eqs. (16) and (18):

$$\tau_{ij} = \bar{\tau}_{ij} a_j, \qquad \nu_{ij} = \bar{\nu}_{ij} a_i a_j \tag{19}$$

This shows that $\bar{\tau}_{ij} a_j$ and $\bar{\nu}_{ij} a_i a_j$ are constant even though a_i is arbitrary. Substituting Eq. (19) into Eq. (13) leads to Eq. (17), where we obtain

$$\bar{\boldsymbol{K}} = \boldsymbol{K} \tag{20}$$

Namely, identified stiffness matrix is unique.

It was stated that the uniqueness of the mass matrix depends on the constraints. Therefore, we can confirm the uniqueness of the stiffness matrix by checking constraints. The symmetry requirement of the stiffness matrix, mode orthogonal constraints and the dynamic equation are selected as constraints. The modal matrix satisfies the dynamic equation

$$\boldsymbol{K} \boldsymbol{\phi} = \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \tag{21}$$

If another solution $\bar{\boldsymbol{K}}$ exists which satisfies the dynamic equation, we obtain

$$\bar{\boldsymbol{K}} \bar{\boldsymbol{\phi}} = \boldsymbol{M} \bar{\boldsymbol{\phi}} \boldsymbol{\Omega}^2 \tag{22}$$

The following equation is obtained by substituting Eq. (7) into Eq. (22):

$$\bar{\boldsymbol{K}} \boldsymbol{\phi} \boldsymbol{A} = \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \boldsymbol{A} \tag{23}$$

Postmultiplying Eq. (23) by $\boldsymbol{A}^{-1}$ gives

$$\bar{\boldsymbol{K}} \boldsymbol{\phi} = \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \tag{24}$$

From Eqs. (21) and (24), we obtain

$$\bar{\boldsymbol{K}} = \boldsymbol{K} \tag{25}$$

Therefore, mode normalization gives no effect on the constraints of the dynamic equation for stiffness matrix identification.

We can reach the same result for the modal stiffness and mode orthogonal constraints. Premultiplying Eq. (21) by $\boldsymbol{\phi}^T$, we obtain

$$\boldsymbol{\phi}^T \boldsymbol{K} \boldsymbol{\phi} = \boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \tag{26}$$

In a similar way, another solution $\bar{\boldsymbol{K}}$ satisfies the following equation.

$$\bar{\boldsymbol{\phi}}^T \bar{\boldsymbol{K}} \bar{\boldsymbol{\phi}} = \bar{\boldsymbol{\phi}}^T \boldsymbol{M} \bar{\boldsymbol{\phi}} \boldsymbol{\Omega}^2 \tag{27}$$

Substituting Eq. (7) into Eq. (27) gives

$$\boldsymbol{A} \boldsymbol{\phi}^T \bar{\boldsymbol{K}} \boldsymbol{\phi} \boldsymbol{A} = \boldsymbol{A} \boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \boldsymbol{A} \tag{28}$$

Premultiplying and postmultiplying Eq. (28) by $\boldsymbol{A}^{-1}$, we obtain

$$\boldsymbol{\phi}^T \bar{\boldsymbol{K}} \boldsymbol{\phi} = \boldsymbol{\phi}^T \boldsymbol{M} \boldsymbol{\phi} \boldsymbol{\Omega}^2 \tag{29}$$

Equations (26) and (29) show that the identified stiffness matrix is unique.

IV. Conclusions

This Note describes the uniqueness of mass and stiffness matrices in system identification. Mass and stiffness matrices are identified by minimizing the Euclidean norm of these matrices subject to some constraints. The identified mass matrix is not unique when modal mass constraints are used. The cause of this problem is the use of mode shapes in the constraints. Since the mode shape is the ratio of amplitude at each point on a structure, the mode shape is not uniquely defined. Namely, any mode shape can be multiplied by a nonzero constant without changing its physical meaning. Therefore, the mode shapes can be normalized in several different expressions. The modal mass constraints are not the constraint imposed on the mass matrix but on mode shapes. This is the reason why it is not reasonable to use the modal mass constraints to uniquely obtain the identified mass matrix. The only way to identify the unique mass matrix is to exclude the modal mass constraints. The identified stiffness matrix is always unique given preexisting condition that the identified mass matrix is unique even if any normalized mode is used.

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